Why are learned indexes so effective?

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A classical problem in computer science

- Given a set of $n$ sorted input keys (e.g. integers)
- Implement membership and predecessor queries
- Range queries in databases, conjunctive queries in search engines, IP lookup in routers...

$\text{member}(36) = \text{True}$

$\text{predecessor}(50) = 48$
Indexes

key

position

B-tree

2 11 13 15 18 23 24 29 31 34 36 44 47 48 55 59 60 71 73 74 76 88 95

1

n

position
Input data as pairs \((key, position)\)
Input data as pairs \((key, position)\)
Learned indexes

Black-box trained on a dataset of pairs (key, pos)
\[ \mathcal{D} = \{(2,1), (11,2), \ldots, (95,n)\} \]

Binary search in position \(-\epsilon, \text{position} + \epsilon\) (approximate)

e.g. \(\epsilon\) is of the order of 100–1000
The knowledge gap in learned indexes

Practice
Same query time of traditional tree-based indexes
Space improvements of orders of magnitude, from GBs to few MBs

Theory
Same asymptotic query time of traditional tree-based indexes
Same asymptotic space occupancy of traditional tree-based indexes
PGM-index: An optimal learned index

1. Fix a max error $\varepsilon$, e.g. so that keys in $[pos - \varepsilon, pos + \varepsilon]$ fit a cache-line
2. Find the smallest Piecewise Linear $\varepsilon$-Approximation (PLA)
3. Store triples ($first$ $key$, $slope$, $intercept$) for each segment
What is the space of learned indexes?

• Space occupancy $\propto$ Number segments
• The number of segments depends on
  • The size of the input dataset
  • How the points \((key, pos)\) map to the plane
  • The value $\varepsilon$, i.e. how much the approximation is precise
Model and assumptions

- Consider gaps $g_i = k_{i+1} - k_i$ between consecutive input keys.
- Model the gaps as positive iid rvs that follow a distribution with finite mean $\mu$ and variance $\sigma^2$. 
The main result

**Theorem.** If $\varepsilon$ is sufficiently larger than $\sigma/\mu$, the expected number of keys covered by a segment with maximum error $\varepsilon$ is

$$K = \frac{\mu^2}{\sigma^2} \varepsilon^2$$

and the number of segments on a dataset of size $n$ is

$$\frac{n}{K}$$

with high probability.
The main consequence

The PGM-index achieves the same asymptotic query performance of a traditional $\varepsilon$-way tree-based index while improving its space from $\Theta(n/\varepsilon)$ to $O(n/\varepsilon^2)$

 Learned indexes are provably better than traditional indexes

(note that $\varepsilon$ is of the order of 100-1000)
Sketch of the proof

1. Consider a segment on the stream of random gaps and the two parallel lines at distance $\varepsilon$.
2. How many steps before a new segment is needed?
Sketch of the proof (2)

3. A discrete-time random walk, iid increments with mean $\mu$

4. Compute the expectation of
   
   $$i^* = \min\{i \in \mathbb{N} \mid (k_i, i) \text{ is outside the red strip}\}$$
   
   i.e. the Mean Exit Time (MET) of the random walk

5. Show that the slope $m = 1/\mu$ maximises $E[i^*]$, giving $E[i^*] = (\mu^2/\sigma^2) \varepsilon^2$
Simulations

1. Generate $10^7$ random streams of gaps according to several probability distributions

2. Compute and average
   I. The length of a segment found by the algorithm that computes the smallest PLA, adopted in the PGM-index
   II. The exit time of the random walk
Simulations of \((\mu^2 / \sigma^2)\varepsilon^2\)  

- **Pareto** \(k = 3, \alpha = 3\)  
- **Lognormal** \(\mu = 1, \sigma = 0.5\)

OPT = Average segment length in a PGM-index  
MET = Mean exit time of the random walk

⇒ Both OPT and MET agree on the slope \(1/\mu\), but OPT is more robust

More distributions in the paper
Stress test of “\( \varepsilon \) sufficiently larger than \( \sigma/\mu \)”
Conclusions

• No theoretical grounds for the efficiency of learned indexes was known
• We have shown that on data with iid gaps, the mean segment length is $\Theta(\varepsilon^2)$
• The PGM-index takes $O(n/\varepsilon^2)$ space w.h.p., a quadratic improvement in $\varepsilon$ over traditional indexes ($\varepsilon$ is usually of the order of 100–1000)

• Open problems:
  1. Do the results still hold without the iid assumption on the gaps?
  2. Is the segment found by the optimal algorithm adopted in the PGM-index a constant factor longer than the one found by the random walker?